Polynomial Functions and Their Graphs

In this section we begin the study of functions defined by polynomial expressions. Polynomial and rational functions are the most common functions used to model data, and are used extensively in mathematical models of production costs, consumer demands, wildlife management, biological processes, and many other scientific studies. Using these functions and their graphs, predictions regarding future trends can be made.

Polynomial Functions and Their Graphs:

Before we start looking at polynomials, we should know some common terminology.

Definition: A **polynomial of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x^1 + a_0$$

where $a_n \neq 0$. The numbers $a_0, a_1, a_2, ..., a_n$ are called the **coefficients** of the polynomial. The number a_0 is the **constant coefficient** or **constant term**. The number a_n, the coefficient of the highest power is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

Notice that a polynomial is usually written in descending powers of the variable, and the degree of a polynomial is the power of the leading term. For instance

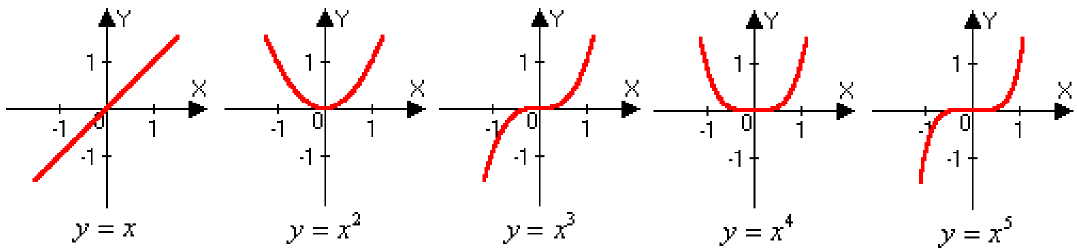
$$P(x) = 4x^3 - x^2 + 5$$

is a polynomial of degree 3. Also, if a polynomial consists of just a single term, such as $Q(x) = 7x^4$, then it is called a **monomial**.

Graphs of Polynomials:

Polynomials of degree 0 and 1 are linear equations, and their graphs are straight lines. Polynomials of degree 2 are quadratic equations, and their graphs are parabolas. As the degree of the polynomial increases beyond 2, the number of possible shapes the graph can be increases. However, the graph of a polynomial function is always a smooth continuous curve (no breaks, gaps, or sharp corners).

Monomials of the form $P(x) = x^n$ are the simplest polynomials.



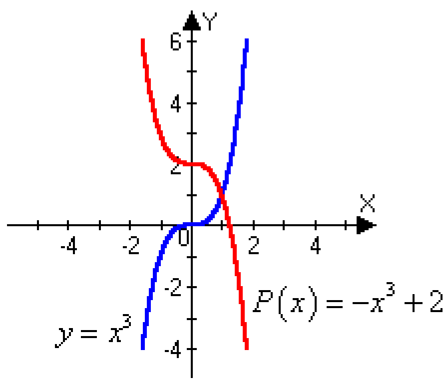
As the figure suggest, the graph of $P(x) = x^n$ has the same general shape as $y = x^2$ when n is even, and the same general shape as $y = x^3$ when n is odd. However, as the degree n becomes larger, the graphs become flatter around the origin and steeper elsewhere.

Transformations of Monomials:

When graphing certain polynomial functions, we can use the graphs of monomials we already know, and transform them using the techniques we learned earlier.

Example 1: Sketch the graph of the function $P(x) = -x^3 + 2$ by transforming the graph of an appropriate function of the form $y = x^n$. Indicate all *x*- and *y*-intercepts on the graph.

Solution: Based on the transformation techniques, we know the graph of $P(x) = -x^3 + 2$ is the reflection of the graph of $y = x^3$ in the *x*-axis, shifted vertically up 2 units. Thus,



Most polynomial functions cannot be graphed using transformations though. For instance in the polynomial function

$$R(x) = 5x^3 - 2x^2 + 1,$$

we cannot determine easily what function's graph we should perform transformations on to graph $R(x)$. Therefore, we will need a new method for finding the graphs of more complex polynomials.

End Behavior of Polynomials:

The **end behavior** of a polynomial is a description of what happens as *x* becomes large in the positive or negative direction. To describe end behavior, we use the following notation:

$x \to \infty$	means	"*x* becomes large in the positive direction"
$x \to -\infty$	means	"*x* becomes large in the negative direction"

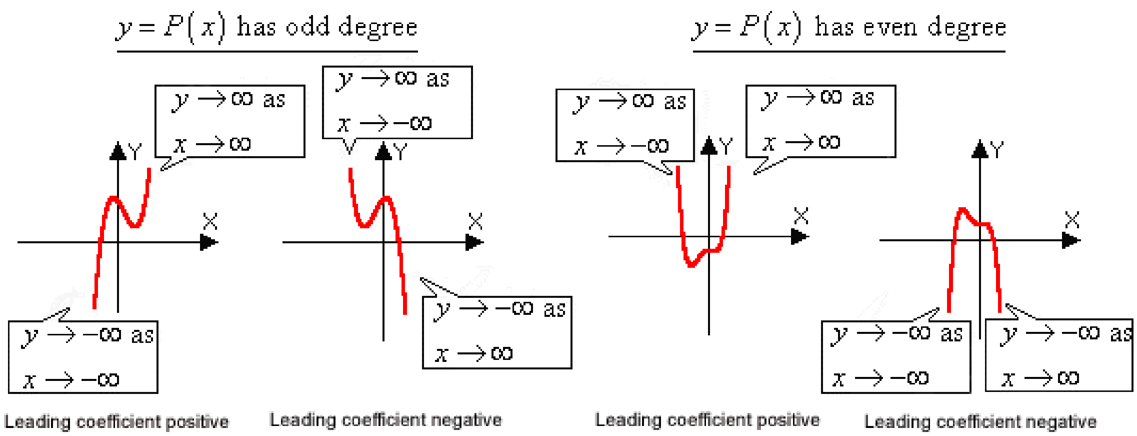
For example, the monomial $y = x^3$ has the end behavior

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

The end behavior of a polynomial graph is determined by the term of highest degree. For instance, the polynomial $f(x) = 3x^5 - 4x^2 + 2$ has the same end behavior as $f(x) = 3x^5$ because both are polynomials of degree 5.

End Behavior of Polynomials

The polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ has the same end behavior as the monomial $Q(x) = a_n x^n$, so its end behavior is determined by the degree n and the sign of the leading coefficient a_n.



Example 2: Determine the end behavior of the polynomial $Q(x)=6x^6-4x^4+2x-3$.

Solution: Since Q has even degree and positive leading coefficient, it has the following end behavior:

$$y\to\infty \quad \text{as} \quad x\to\infty \qquad \text{and} \qquad y\to\infty \quad \text{as} \quad x\to-\infty$$

Using Zeros to Graph Polynomials:

Definition: If P is a polynomial and c is a number such that $P(c)=0$, then we say that c is a **zero** of P. The following are equivalent ways of saying the same thing.

1. c is a **zero** of P
2. $x=c$ is a **root** of the equation $P(x)=0$
3. $x-c$ is a factor of $P(x)$

When graphing a polynomial, we want to find the roots of the polynomial equation $P(x)=0$. To do this, we factor the polynomial and then use the Zero-Product Property (Section 3.3). Remember that if $P(c)=0$, then the graph of $y=P(x)$ has an x-intercept at $x=c$, so the x-intercepts of the graph are the zeros of the function.

Example 3: Find the zeros of the polynomial $R(x)=x^2-7x+12$.

Solution:

Step 1: First we must factor R to get

$$R(x)=(x-4)(x-3)$$

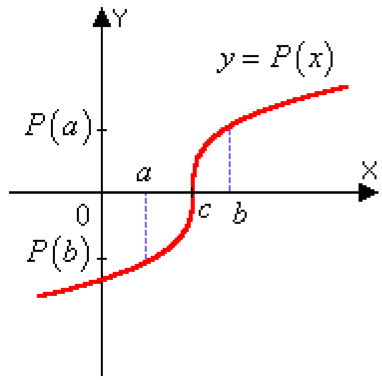
Step 2: Since $x-4$ is a factor of $R(x)=x^2-7x+12$, 4 is a zero of R, and since $x-3$ is a factor of $R(x)=x^2-7x+12$, 3 is a zero of R.

The following theorem and its consequences will be used to help us graph polynomials.

Intermediate Value Theorem for Polynomials:

If P is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b for which $P(c)=0$.

The figure below graphically demonstrates this theorem.



One important consequence of this theorem is that between any two successive zeros, the values of a polynomial are either all positive or all negative. That is, between two successive zeros the graph of a polynomial lies *entirely above or entirely below* the *x*-axis.

So, to sketch the graph of *P*, we first find all the zeros of *P*. Then we choose **test points** between (and to the right and left of) successive zeros to determine whether $P(x)$ is positive or negative on each interval determined by the zeros.

Guidelines for Graphing Polynomial Functions:

1. **Zeros:** Factor the polynomial to find all its real zeros; these are the *x*-intercepts of the graph.

2. **Test Points:** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the *x*-axis on the intervals determined by the zeros. Include the *y*-intercept in the table

3. **End Behavior:** Determine the end behavior of the polynomial.

4. **Graph:** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

Example 4: Sketch the graph of the function $P(x)=(x+1)(x-2)^2$. Make sure your graph shows all intercepts and exhibits the proper end behavior.

Solution:

Step 1: First we must find all the real zeros of $P(x)$. Since $P(x)=(x+1)(x-2)^2$ is already factored, it is easy to see the zeros are $x=-1$ and $x=2$. Therefore, the *x*-intercepts are $x=-1$ and $x=2$.

Step 2: Now we will make a table of values of $P(x)$, making sure we choose test points between (and to the left and right of) successive zeros, and include the *y*-intercept, $P(0)=4$.

	x	$P(x)$	
Testpoint $\rightarrow$	-2	-16	
	-1	0	$\leftarrow$ x-intercept
Testpoint $\rightarrow$	0	4	$\leftarrow$ y-intercept
	2	0	$\leftarrow$ x-intercept
Testpoint $\rightarrow$	3	4	

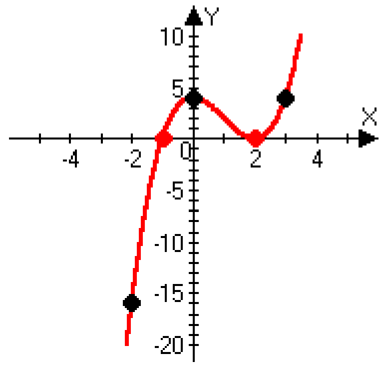
Step 3: Next we determine the end behavior. If we expand *P*, we get $P(x)=x^3-2x^2+4$. Since *P* is of odd degree (degree 3) and its leading coefficient is positive, it has the following end behavior:

$$y\to\infty \quad \text{as} \quad x\to\infty \qquad \text{and} \qquad y\to-\infty \quad \text{as} \quad x\to-\infty$$

Example 4 (Continued):

Solution:

Step 4: Finally, we plot the points from the table and connect the points by a smooth curve to complete the graph.



$P(x)=(x+1)(x-2)^2$

Example 5: Sketch the graph of the function $Q(x)=x^4-4x^2$. Make sure your graph shows all intercepts and exhibits the proper end behavior.

Solution:

Step 1: In order to find all the real zeros of $Q(x)$, we must first factor it completely.

$$\begin{aligned} Q(x) &= x^4-4x^2 \\ &= x^2(x^2-4) && \text{Factor } x^2 \\ &= x^2(x+2)(x-2) && \text{Difference of Squares} \end{aligned}$$

Step 2: Since $Q(x)=x^2(x+2)(x-2)$ the zeros are $x=0$, $x=-2$ and $x=2$. Thus, the x-intercepts are $x=0$, $x=-2$ and $x=2$.

Example 5 (Continued):

Solution:

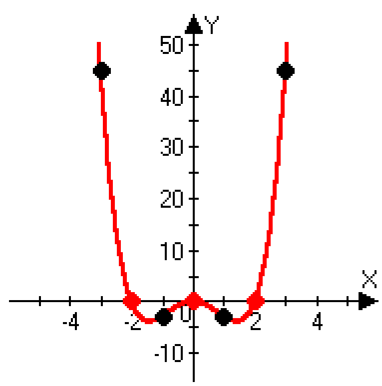
Step 3: Now we will make a table of values of $Q(x)$, making sure we choose test points between (and to the left and right of) successive zeros, and include the *y*-intercept, $Q(0)=0$.

	x	$Q(x)$	
Testpoint $\rightarrow$	-3	45	
	-2	0	$\leftarrow$ x-intercept
Testpoint $\rightarrow$	-1	-3	
	0	0	$\leftarrow$ x-intercept, y-intercept
Testpoint $\rightarrow$	1	-3	
	2	0	$\leftarrow$ x-intercept
Testpoint $\rightarrow$	3	45	

Step 4: Next we determine the end behavior. Since *Q* is of even degree (degree 4) and its leading coefficient is positive, it has the following end behavior:

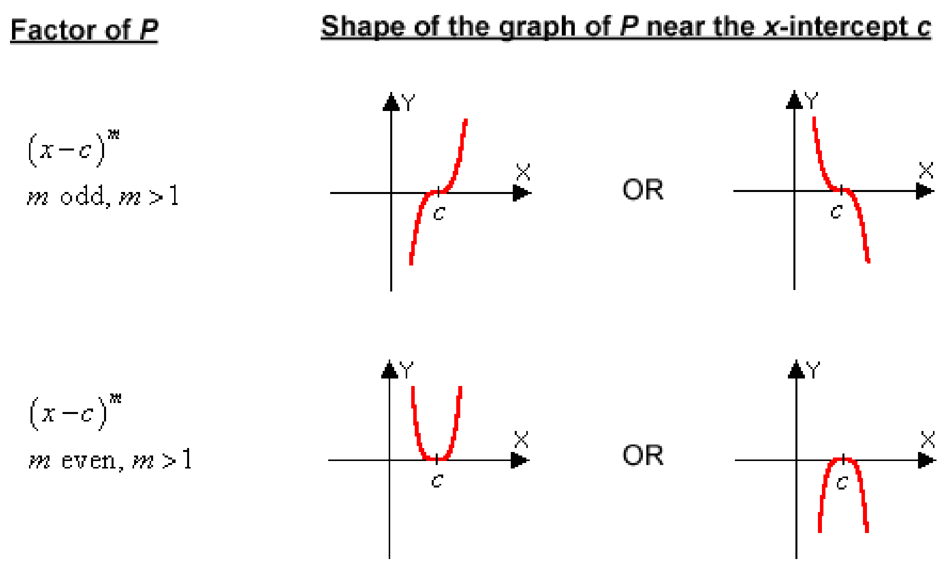
$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to -\infty$$

Step 5: Finally, we plot the points from the table and connect the points by a smooth curve to complete the graph.



$Q(x) = x^4 - 4x^2$

By examining Example 4 and Example 5, notice that when c is a zero of P, and the corresponding factor $(x-c)$ occurs *exactly* m times in the factorization of P, the graph crosses the x-axis at c if m is odd and does not cross the x-axis if m is even.



Example 6: Sketch the graph of the function $R(x)=-(x-2)^2(x+1)^2(x+2)$. Make sure your graph shows all intercepts and exhibits the proper end behavior.

Solution:

Step 1: The real zeros of $R(x)=-(x-2)^2(x+1)^2(x+2)$ are $x=2$, $x=-1$ and $x=-2$. These are the x-intercepts of the graph.

Example 6 (Continued):

Solution:

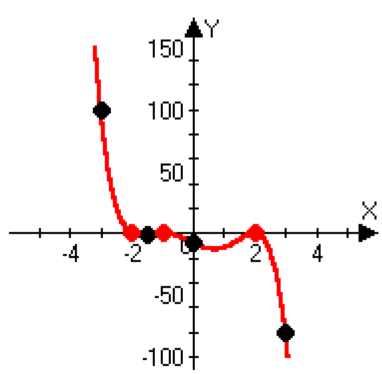
Step 2: Now we will make a table of values of $R(x)$, making sure we choose test points between (and to the left and right of) successive zeros, and the *y*-intercept, $R(0) = -8$.

x	$P(x)$
-3	100
-2	0
-1.5	-1.53
-1	0
0	-8
2	0
3	-80

Step 3: Since *R* is of odd degree (degree 5) and its leading coefficient is negative, it has the following end behavior:

$$y \to -\infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to -\infty$$

Step 4: Finally, we plot the points from the table and connect the points by a smooth curve to complete the graph.



$R(x) = -(x-2)^2(x+1)^2(x+2)$