Polynomial and Rational Inequalities

This section will explore how to solve inequalities that are either in rational or polynomial form.

Example 1. **Solve the equation** $x^2 < x + 6$ **and graph the solution on a number line.**

Step 1. **Write the equation in standard form.**

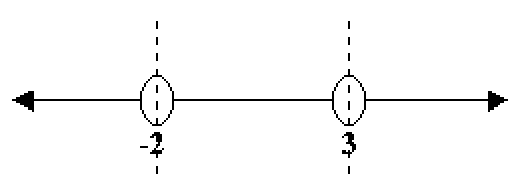
$$x^2 < x + 6$$
$$x^2 - x - 6 < x + 6 - x - 6$$
$$x^2 - x - 6 < 0$$

Step 2. **Find the zeros of the equation (the values of *x* that make the equation equal to zero).**

$$x^2 - x - 6 < 0$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x - 3 = 0 \quad or \quad x + 2 = 0$$
$$x = 3 \quad or \quad x = -2$$

Step 3. **Organize data.**

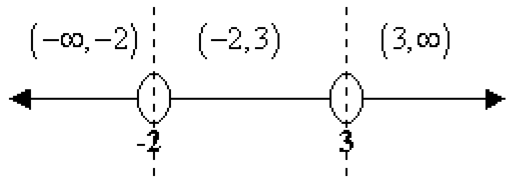
The values found in Step 2, (3, 0) **and** (-2, 0) **are plotted on a number line and the number line is divided into sections (intervals) with boundaries passing through these points:**



These values are known as critical points because the boundaries of any intervals pass through them.

Example 1 (Continued):

Step 4. **Test each interval for its sign values.**



To find the sign value of each interval, select any point within the interval (EXCEPT THE CRITICAL POINTS) and substitute the value for *x* in the factored form of the polynomial.

Test point for the interval (-∞, -2)**:** ***x* = -5**

$$(x-3)(x+2)$$
$$(-5-3)(-5+2)$$
$$(-8)(-3)=24$$

The product found was positive, therefore the interval is also positive.

Test point for the interval (-2, 3)**:** ***x* = 0**

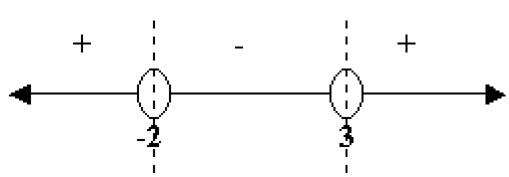
$$(x-3)(x+2)$$
$$(0-3)(0+2)$$
$$(-3)(2)=-6$$

The product found was negative, therefore the interval is also negative.

Test point for the interval (3, ∞)**:** ***x* = 5**

$$(x-3)(x+2)$$
$$(5-3)(5+2)$$
$$(2)(7)=14$$

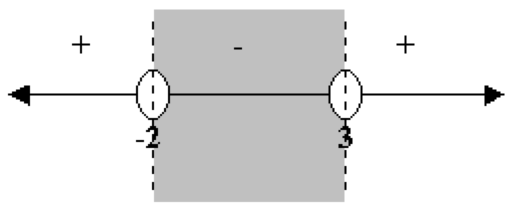
The product found was positive, therefore the interval is also positive.



Example 1 (Continued):

Step 5. **Graph.**

Since the problem desires values that are less than zero they will be negative in nature. Since the interval (-2, 3) **contains the only negative results, this interval and all of its values are the only solution. Note that the values of** x = -2 **and** x = 3 **are not part of the solution because these values make the equation equal to zero when they are used, not less than zero. The graph of the solution is:**



Example 2. **Solve** $\frac{2x-7}{x-5} \leq 3$ **and graph the solution.**

Step 1. **Write the equation in standard form.**

$$\frac{2x-7}{x-5} \leq 3$$

$$\frac{2x-7}{x-5} - 3 \leq 3 - 3$$

$$\frac{2x-7}{x-5} - 3 \leq 0$$

$$\frac{2x-7}{x-5} - \left(\frac{3}{1}\right)\left(\frac{x-5}{x-5}\right) \leq 0$$

$$\frac{2x-7}{x-5} - \frac{(3x-15)}{x-5} \leq 0$$

$$\frac{2x-7-3x+15}{x-5} \leq 0$$

$$\frac{-x+8}{x-5} \leq 0$$

Example 2 (Continued):

Step 2. **Find the critical points.**

Note that in addition to the values that make the equation equal to zero, in rational expressions a critical point will involve those values that cannot be used (i.e. the values that would make the denominator zero).

$$\frac{-x+8}{x-5} \le 0$$

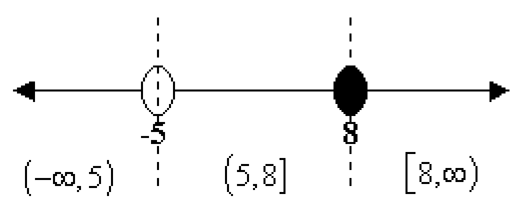
$$-x+8=0 \quad or \quad x-5=0$$

$$8=x \quad or \quad x=5$$

$\therefore$ **The critical points for this equation are** (8, 0) **and** (5, 0)

Step 3. **Organize data.**

Since the equation will allow zero to be a solution, the critical point (8, 0) **is part of the solution set, whereas the point** (5, 0) **is not. The reason is the substitution of** x = 5 **would result in a division by zero. As can be seen on the number line** (-∞, 5)**,** (5, 8] **and** [8, ∞) **are the intervals to be inspected.**



Example 2 (Continued):

Step 4. **Test the intervals for the sign values.**

Test point for the interval $(-\infty, 5)$**:** **x = 0**

$$\frac{-x+8}{x-5} < 0$$
$$\frac{-(0)+8}{0-5} < 0$$
$$-\frac{8}{5} < 0$$

This value is negative so the intervals values are also negative. Therefore this interval is part of the solution.

Test point for the interval $(5, 8]$**:** **x = 6**

$$\frac{-x+8}{x-5} < 0$$
$$\frac{-(6)+8}{6-5} < 0$$
$$\frac{2}{1} < 0$$

This value is positive so the intervals values are also positive. Therefore this interval is not part of the solution.

Test point for the interval $[8, \infty)$**:** **x = 10**

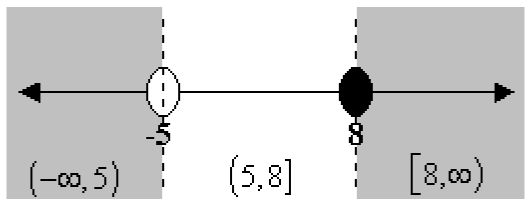
$$\frac{-x+8}{x-5} < 0$$
$$\frac{-(10)+8}{10-5} < 0$$
$$-\frac{2}{5} < 0$$

This value is negative so the intervals values are also negative. Therefore this interval is part of the solution.

Example 2 (Continued):

Step 5. Graph the solution.

From Step 4 it is determined that the graph and solutions for the equation are:



This may be written in inequality notation as $-\infty < x < 5$ **or** $x \geq 8$ **and in interval notation as** $(-\infty, 5) \cup [8, \infty)$**.**